Two Reconstruction Algorithms of Non Gaussian Processes on the Output of a Polynomial Converter

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Abstract. Two extrapolation algorithms are investigated for reconstruction procedures of non Gaussian processes. We investigate the process on the output of polynomial converter driven by Gaussian process. The optimal reconstruction algorithm is investigated on the basis of the conditional mean rule with help of cumulant functions. In this case the error reconstruction function depends on the given samples. Another algorithm is non optimal, because the reconstruction operation is realized by using of a covariance function of the output process only. The case with the polynomial of the third order is investigated in detail.

1 Introduction

The problem of reconstruction of a signal that passes through determinate samples has been investigated since XIX century. The classical Sampling Theorem is usually associated with the names of Whittaker, Kotelnikov and Shannon (or WKS theorem), and it has been proved for deterministic functions with the limited spectrum. This classical theorem has been generalized on stochastic stationary processes by A. Balakrishnan [1]. Following Balakrishnan´s theorem [1] all types of random stationary processes with a limited power spectrum can be reconstructed without error by the unique reconstruction function sin x / x when the number of samples is equal to infinity. But some important characteristics like the Probability Density Function (pdf), limit number of samples and high order moments are not mentioned in this theorem. In fact, there are some publications where recommendations of Balakrishnan´s theorem are applied for the cases with limited number of samples and with an arbitrary pdf of processes.

In the present paper we analyze the statistical description of Sampling-Reconstruction Procedure (SRP) of non Gaussian process. This process is formed on the output of a converter with a non linear polynomial characteristic. The process in the input is Gaussian Markovian. We take an extrapolation case (only one sample).

The first algorithm is optimal. This algorithm is based on the conditional mean rule [2]. The evaluation of a reconstructed realization is formed by the conditional mean. This evaluation provides a minimum mean square error which

© G. Sidorov, M. Aldape, M. Martinez, S. Torres. (Eds.)
Advances in Computer Science and Engineering.
Research in Computing Science 42, 2009, pp. 287-295

Received 01/04/09
Accepted 27/04/09
Final version 07/05/09

is described by the conditional variance. Using the method suggested in [3], we obtain the conditional mean and the conditional variance of the output non Gaussian process on the bases of conditional cumulant functions [4].

The second algorithm is non optimal, because the reconstruction operation is based on the covariance function of the output process only. In other words, the extrapolation procedure of non Gaussian process is formed like the extrapolation function of Gaussian process does.

We analyze two mentioned variants for the polynomial characteristic of the third order in detail. The conclusion is: it is necessary to take into account the pdf of sampled process in the statistical SRP description of random processes.

2 Features of the Process on the Output of a Polynomial Converter

In a polynomial case, in order to find the right methodology that describes the properties of the non-linear converter, we assume the non-linearity expressed by the formula:

$$\eta(t) = g[x(t)] = a_o + a_1 x(t) + a_2 x^2(t) + ... + a_n x^n(t) \tag{1}$$

where $a_i (i = 0, 1, 2, ..., n)$ are constants.

We choose three variants transfer functions of third order:

$$\eta(t) = g(\xi) = \xi^3 \tag{2}$$

$$\eta(t) = g(\xi) = \xi^3 + 2\xi \tag{3}$$

$$\eta(t) = g(\xi) = 1.6\xi^3 + 5.1\xi \tag{4}$$

they are shown in the Fig. 1.

The expressions (2) – (4) have the unique inverse functions $\xi(t) = h(\eta(t))$.

We suppose that the process $\xi(t)$ is Gaussian Markov with the covariance function:

$$k(\tau) = \sigma^2 \exp(-\alpha \mid \tau \mid) \tag{5}$$

and the mathematical expectation is zero.

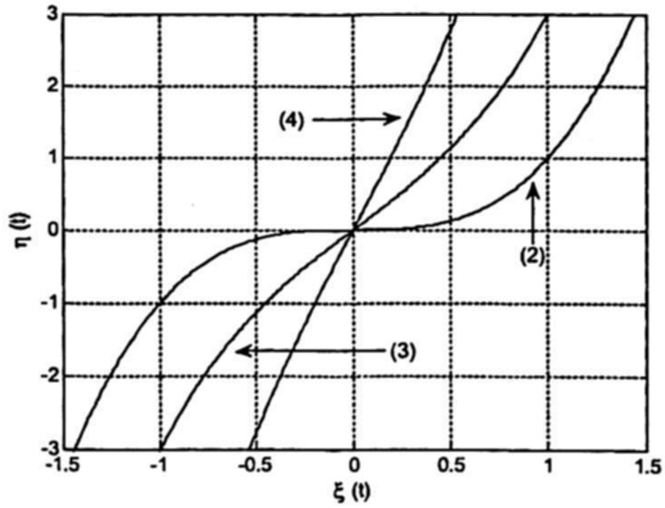

Fig. 1. Polynomial Non-linearity

We put $\sigma^2 = 1$. Firstly we determinate the unconditional characteristics of the process $\eta(t)$.For getting an one-dimensional pdf for the transfer functions, we use the next methodology [3]:

$$w(\eta) = w(h(\eta))\frac{1}{\mid \frac{d\eta}{d\xi} \mid} = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}\xi^2\}\frac{1}{\mid \frac{d\eta}{d\xi} \mid} \tag{6}$$

The graphs of these functions are presented in Fig. 2. Knowing pdf (6), we can calculate all required moments of the output process. It is clear that the mathematical expectation $\langle \eta(t) = 0 \rangle$ for all types of non linearity, formula (7). The variance calculations give the values: 15, 31 and 113.35 for the cases (2), (3) and (4) respectively, formula (8).

$$m_n = \langle \eta \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}\xi^2\} \cdot g(\xi) \cdot d\xi \tag{7}$$

$$\sigma_\eta^2 = \langle \eta^2 \rangle = \langle \eta \rangle^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}\xi^2\} \cdot [g(\xi)]^2 \cdot d\xi \tag{8}$$

Rewriting (1) for two time moments t and $t+\tau$, multiplying both expressions and applying the average operation, we can find the covariance function of the output process, restricting by the order 3, by the following formula [4]:

$$K_\eta[\tau] = \nu_1^2 K_\xi[\tau] + \frac{\nu_2^2}{2!}K_\xi^2[\tau] + \frac{\nu_3^2}{3!}K_\xi^3[\tau] \tag{9}$$

where the coefficients ν_n are determined by [4]:

$$\nu_n(m_\xi, \sigma_\xi^2) = n! a_n \tag{10}$$

Although the extrapolation algorithm only depends on the sample $\eta(T_n)$. The output covariance function is showed in Fig. 3.

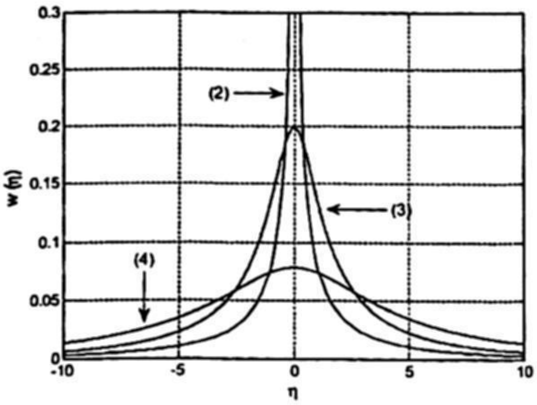

Fig. 2. Probability Density Function

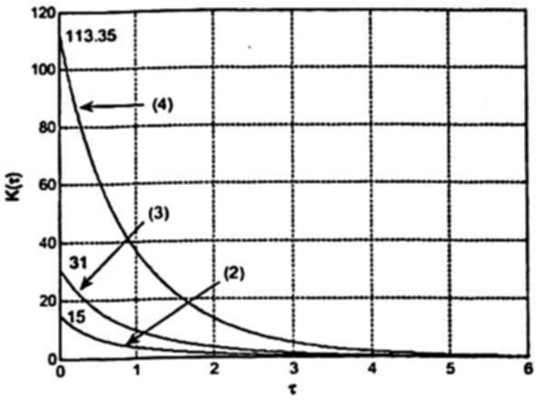

Fig. 3. Covariance output function

In Fig. 2 we can see the dispersion of the possible values. The covariance function grows according the dispersion on the fpd.

3 The Optimal Extrapolation Reconstruction Algorithm

Let us fix a set of samples Ξ =$[\xi_1, \xi_2, ..., \xi_n]$. Owing to the unique inverse functions $\xi(t) = h(\eta(t))$ we find the corresponding set η =$[\eta_1, \eta_2, ..., \eta_n]$. Then, we can apply the statistical conditional average operation to both parts of the expression (1). In result we have the expression for the reconstruction function:

$$\tilde{m}_1^{\eta}(t) = a_0 + a_1\tilde{m}_1^{\xi}(t) + ... + a_n\tilde{m}_n^{\xi}(t) \tag{11}$$

where $\tilde{m}_1^{\eta}(t)$ is the conditional mathematical expectation of the output process, $\tilde{m}_i^{\eta}(t)(i = 1, ..., n)$ are the conditional moment functions of the order i of the input process.

Let us calculate the square of both parts of the expression (1) and fulfill the conditional average operation, this yield:

$$\tilde{m}_2^{\eta}(t) = a_0^2 + a_1^2\tilde{m}_2^{\xi}(t) + ... + a_n^2\tilde{m}_{2n}^{\xi}(t) + 2a_0a_1\tilde{m}_1^{\xi}(t) + ... + a_{n-1}a_n\tilde{m}_{2n-1}^{\xi}(t) \tag{12}$$

Knowing (11) and (12) we can find the required conditional variance or the error reconstruction function:

$$\tilde{\sigma}_{\eta}^2(t) = \tilde{m}_2^{\eta}(t) - [\tilde{m}_1^{\eta}(t)]^2 \tag{13}$$

The equations (11) – (13) show that the reconstruction function and the error reconstruction function require the high order conditional moments on the input. In order to determine the output moment functions we can apply some connection expressions between moments and cumulants [4,5]. The Gaussian pdf is described by the two first cumulants. Then, the relations between the conditional moments $\tilde{m}_1^\xi(t)$ on the input and the conditional first $\tilde{k}_1$ and second cumulants $\tilde{k}_2$ are:

$$\begin{aligned}
\tilde{m}_1^\xi(t) &= \tilde{k}_1 \\
\tilde{m}_2^\xi(t) &= \tilde{k}_2 + \tilde{k}_1^2 \\
\tilde{m}_3^\xi(t) &= 3\tilde{k}_2\tilde{k}_1 + \tilde{k}_1^3 \\
\tilde{m}_4^\xi(t) &= 3\tilde{k}_2^2 + 6\tilde{k}_2\tilde{k}_1^2 + \tilde{k}_1^4 \\
\tilde{m}_5^\xi(t) &= 15\tilde{k}_2^2\tilde{k}_1 + 10\tilde{k}_2\tilde{k}_1^3 + \tilde{k}_1^5 \\
\tilde{m}_6^\xi(t) &= 15\tilde{k}_2^3 + 45\tilde{k}_2^2\tilde{k}_1^2 + 15\tilde{k}_2\tilde{k}_1^4 + \tilde{k}_1^5
\end{aligned} \tag{14}$$

The conditional mathematic expectation $\tilde{k}_1$ and the conditional variance $\tilde{k}_2$ are expressed by the formulas [6]:

$$\tilde{k}_1 = \tilde{m}^\xi(t) = m^\xi + \sum_{i=1}^{N}\sum_{j=1}^{N} K_\xi(t - T_i)a_{ij}[\xi(T_j) - m^\xi(T_j)] \tag{15}$$

$$\tilde{k}_2 = \tilde{\sigma}_\xi^2(t) = \sigma_\xi^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} K_\xi(t - T_i)a_{ij}K_\xi(T_j - t) \tag{16}$$

a_{ij} is an element of the inverse covariance matrix:

$$| \, a_{ij} \, | = | \, K_\xi(T_i, T_i) \, |^{-1} \tag{17}$$

From (11) - (16) one can see that the conditional variance $\tilde{\sigma}_\eta^2(t)$ depends on the values of samples, but in a Gaussian case it does not.

We consider a particular case when the set of samples has only one term . Let us choose the following number and values of input $\xi(T)$ and output $\eta(T)$ samples, on Table 1.

Table 1. Input and Output samples

	$\xi(T)$	$\eta(T) = \xi^3$	$\eta(T) = \xi^3 + 2\xi$	$\eta(T) = 1.6\xi^3 + 5.1\xi$
a)	0.4	0.064	0.864	2.142
b)	0.7	0.343	1.743	4.119
c)	1	1	3	6.7
d)	1.3	2.197	4.797	10.15
e)	1.5	3.375	6.375	13.05

The results of the reconstruction function are presented in Fig. 4.a, 5.a and 6.a for (2), (3) and (4) respectively, we can observe that they have a nonlinear performance. The error reconstruction function is presented in Fig. 4.b, 5.b and 6.b for (2), (3) and (4) respectively, all those curves converge at the value of their variance, they characterize the minimum error for the optimal reconstruction algorithm of the extrapolation type, because of the output process is not Gaussian, the error reconstruction depends on the sample value.

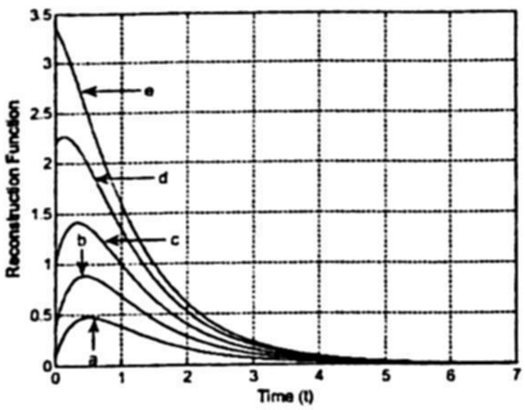

Fig. 4.a. Optimal Reconstruction Function for $\eta(t) = \xi^3$

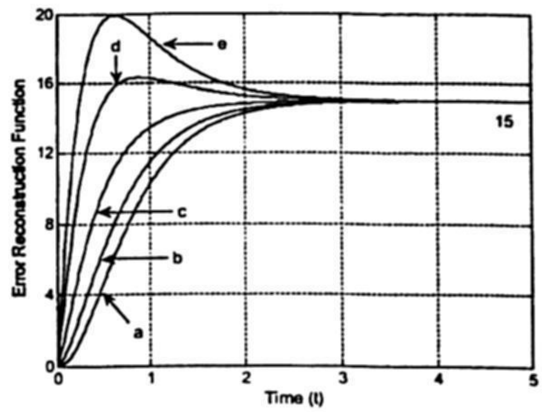

Fig. 4.b. Optimal Error Reconstruction Function for $\eta(t) = \xi^3$

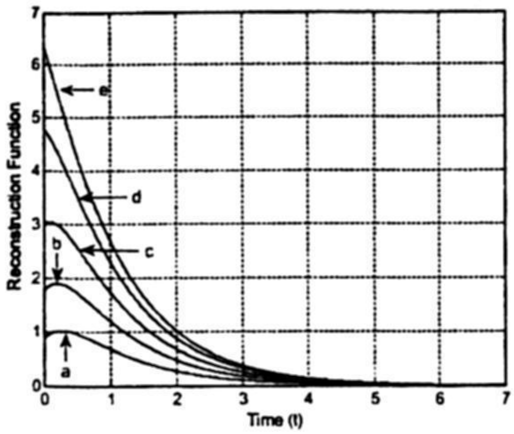

Fig. 5.a. Optimal Reconstruction Function for $\eta(t) = \xi^3 + 2\xi$

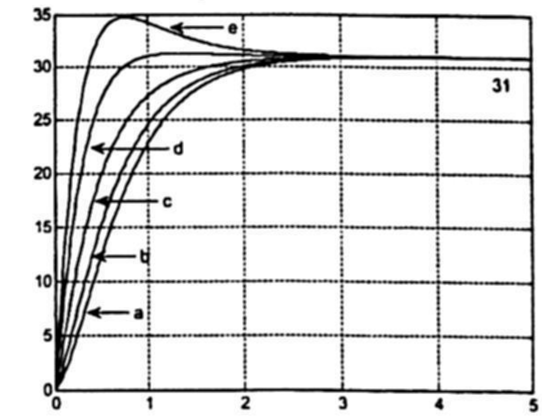

Fig. 5.b. Optimal Error Reconstruction Function for $\eta(t) = \xi^3 + 2\xi$

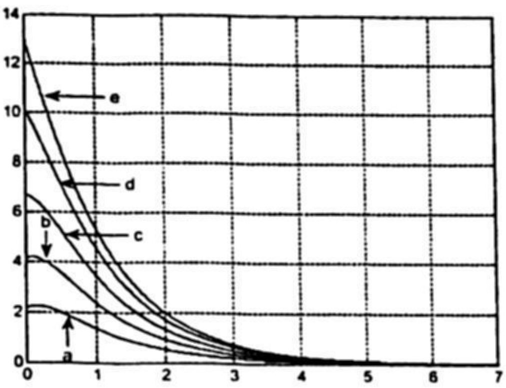

Fig. 6.a. Optimal Reconstruction Function for $\eta(t) = 1.6\xi^3 + 5.1\xi$

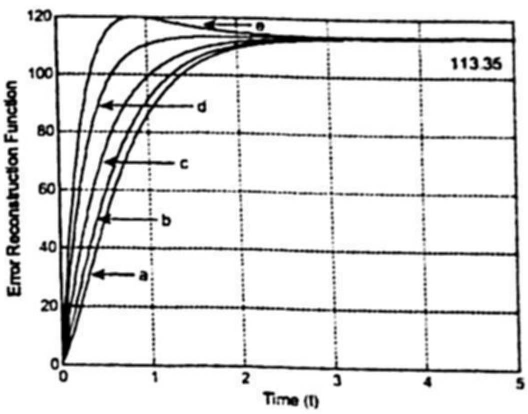

Fig. 6.b. Optimal Error Reconstruction Function for $\eta(t) = 1.6\xi^3 + 5.1\xi$

All those groups of curves have the clear physically significances, and explain the sense of the non Gaussian distribution of the output process.

4 Non Optimal Extrapolation Reconstruction Algorithm

The non optimal algorithm is based on the knowledge of the covariance output function only. We just need the Gaussian approach to describe the reconstruction, so we must to apply the next formulas:

$$\tilde{m}^{\eta}(t) = m^{\eta} + \sum_{i=1}^{N}\sum_{j=1}^{N} K_{\eta}(t - T_i)a_{ij}[\eta(T_j) - m^{\eta}(T_j)] \tag{18}$$

$$\tilde{\sigma}_{\eta}^2(t) = \sigma_{\eta}^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} K_{\eta}(t - T_i)a_{ij}K_{\eta}(T_j - t) \tag{19}$$

when N=1, we have:

$$\tilde{m}^{\eta}(t) = m^{\eta} + \frac{K_{\eta}(t - T_1)}{\sigma_{\eta}^2}[\eta(T_1) - m^{\eta}(T_1)] = \frac{K_{\eta}(t - T_1)}{\sigma_{\eta}^2}[\eta(T_1)] \tag{20}$$

$$\tilde{\sigma}_{\eta}^2(t) = \sigma_{\eta}^2[1 - R_{\eta}^2(t - T_1)] \tag{21}$$

where $R_{\eta}(\tau) = \frac{K_{\eta}(\tau)}{\sigma_{\eta}^2}$

Substituting (5) – (10) on the last formulas, we obtain the reconstruction function that is a group of linear curves; and the error reconstruction function which is only one curve that converge to their variance, as in optimal algorithm.

There is just one curve for each transfer function because this method doesn't take the samples for finding the error reconstruction. The graphics for the reconstruction function are in the Fig. 7, they are the same for (2), (3) and (4) using a) $\xi(t) = \eta(t) = 0.4$, b) $\xi(t) = \eta(t) = 0.7$, c) $\xi(t) = \eta(t) = 1$, d) $\xi(t) = \eta(t) = 1.3$, e) $\xi(t) = \eta(t) = 1.5$; this is because of the lineal method. The graphics of the error reconstruction function are showed in the Fig. 8 for (2), (3) and (4).

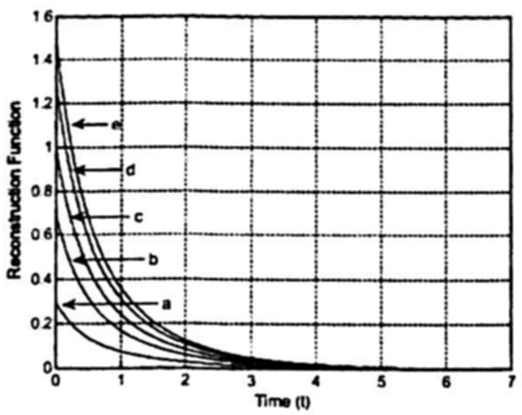

Fig. 7. Non Optimal Reconstruction Function

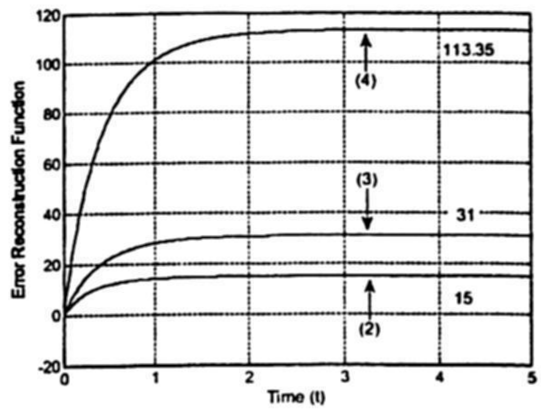

Fig. 8. Non Optimal Error Reconstruction Function

The non optimal reconstruction is characterized by one error curve (Fig. 8). This fact is connected with the Gaussian approximation of the output process. The real situation is another; the non Gaussian pdf of the output process determines absolutely another behavior of the error reconstruction curves. The error reconstruction function depend on the samples, the Fig 4.b, 5.b and 6.b show that the big samples $(\xi(T_1 > 1))$ have bigger error reconstruction function. It means that it is necessary to take into account the pdf of sampled processes.

5 Conclusions

Two different reconstruction algorithms are investigated for output processes of non linear polynomial converters driven by Gaussian Markovian process. Both principal characteristics (reconstruction function and error reconstruction function) are obtained. Comparison of these algorithms shows that it is necessary to take into account pdf of sampled process. The non optimal algorithm gives non correct results. And, the optimal case is an easier methodology to use.

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